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Z_N-Graded Noncommutative Differential Calculus

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We consider the algebra $M_N(C)$ of $N \times N$ matrices as a cyclic quantum plane. We also analyze the coaction of the quantum group $\mathcal{F}$ and the action of its dual quantum algebra $\mathcal{H}$ on it. Then we study the decomposition of $M_N(C)$ in terms of the quantum algebra representations. Finally, we develop the differential algebra of the cyclic group Z_N with $d^N = 0$, where Z_N is viewed as the the subalgebra of diagonal $N \times N$ complex matrices, and treat the particular case $N = 3$.

1. INTRODUCTION

In the last decade, the concept of noncommutative differential geometry [1] has been extensively developed. The simplest example of noncommutative differential geometry based on derivations is given by the Grassmannian of the matrix algebra $\mathcal{M}_N = M_N(C)$ [2]. The matrix algebra $\mathcal{M}_N$ can also be considered as a *cyclic quantum plane* ($q^N = 1$) on which a coaction of quantum group $\mathcal{F}$ and an action of its dual $\mathcal{H}$ are naturaly defined, and the associated Wess–Zumino differential complex is constructed (see ref. 3 and references therein). Moreover, the notion of a graded q-differential algebra with the condition $d^N = 0$ has been recently introduced [4].

The main aim of this work is to study the noncommutative differential geometry of the cyclic group Z_N viewed as the subalgebra $\mathcal{M}_N^{\text{diag}}$ of diagonal matrices of $\mathcal{M}_N$, as an example of Z_N-graded noncommutative differential calculus.

This work is organized as follows: In Section 2 we present the space $M_N(C)$ as a cyclic Manin plane. Then we present the coaction and the action of the quantum group $\mathcal{F}$ and its dual $\mathcal{H}$ on $\mathcal{M}_N$, respectively, and study the reduction of $\mathcal{M}_N$ under the representation of $\mathcal{H}$. In Section 3 we construct

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the noncommutative differential complex of the cyclic group Z_N with a Z_N-graded differential d, i.e., $d^N = 0$, and treat in detail the particular case $N = 3$.

2. THE CYCLIC MANIN PLANE

2.1. $\mathcal{M}_N \equiv M_N(C)$ as a Cyclic Quantum Plane

The algebra of $N \times N$ matrices can be generated by two elements x and y obeying the relations [5]

$$xy = qyx \tag{1}$$

$$x^N = y^N = \mathbf{1} \tag{2}$$

where q denotes a primitive Nth root of unity:

$$q^n = 1, \qquad \sum_{n}^{N-1} q^n = 0 \tag{3}$$

and $\mathbf{1}$ is the $N \times N$ unit matrix.

Explicitly, x and y can be represented by the matrices

$$x = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdots & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdots & \cdot & 0 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^{N-1} \end{pmatrix}$$

We call the algebra generated by elements x and y satisfying the relations (1) and (2) the *cyclic quantum plane* $\mathcal{M}_N \equiv M_N(C)$. As an N^2-dimensional vector space, $\mathcal{M}_N$ is spanned by the following basis:

$$\{\alpha^{rs} = x^r y^s;\ r, s = 0, 1, 2, \ldots, N-1\}$$

and is endowed by the following internal law:

$$\alpha^{rs}.\alpha^{mn} = f^{(rs)(mn)}{}_{(kl)}\alpha^{kl}$$

where $x^r y^s = q^{sr} y^s x^r$ and

$$f^{(rs)(mn)}{}_{(kl)} = q^{-ms}\,\delta^{r+m}{}_k\,\delta^{s+n}{}_l$$

The *noncommutativity* of the elements of $\mathcal{M}_N$ is reflected in the following relation:

$$\alpha^{rs}.\alpha^{mn} = q^{(rn-ms)}\alpha^{mn}\alpha^{rs}$$

We can also equip $\mathcal{M}_N$ with a Lie structure by introducing the following commutation rule:

$$[\alpha^{rs}, \alpha^{mn}] = C^{(rs)(mn)}{}_{(kl)}\,\alpha^{kl}$$

where the structure constants are given by

$$C^{(rs)(mn)}{}_{(kl)} = (q^{-ms} - q^{-nr})\,\delta^{r+m}{}_k\,\delta^{s+n}{}_l$$

Let us define a basis $\{e_{rs}\}$ of Der($\mathcal{M}_N$), i.e., the Lie algebra of derivations (all are inner) of $\mathcal{M}_N$ as follows:

$$e_{rs} = Ad_{\alpha^{\mathrm{rs}}} = [\alpha^{rs}, .]$$

such that

$$e_{rs}(\alpha^{mn}) = [\alpha^{rs}, \alpha^{mn}] = C^{(rs)(mn)}{}_{(kl)}\,\alpha^{kl}$$

and satisfying

$$[e_{rs}, e_{mn}] = C_{(rs)(mn)}{}^{(kl)}\,e_{kl}$$

2.2. The Quantum Group $\mathcal{F}$ and Its Coaction on $\mathcal{M}_N$

Let us construct the matrix quantum group generated by the quantum matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ coacting on the coordinate doublet of the reduced quantum plane by the following left and right coactions:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \delta_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\otimes x + b\otimes y \\ c\otimes x + d\otimes y \end{pmatrix}$$

$$\begin{aligned}(x'' \quad y'') &= \delta_{\mathrm{R}}\,(x \quad y) = (x \quad y) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= (x\otimes a + y\otimes c \quad x\otimes b + y\otimes d)\end{aligned}$$

Imposing that the quantities x', y' (and x'', y'') should satisfy the same relations as x and y, one obtains the following defining relations of the quantum bialgebra $M_q(2; C)$:

$$
\begin{aligned}
ab &= qba \\
ac &= qca \\
ad - da &= (q - q^{-1})bc \\
bc &= cb \\
bd &= qdb \\
cd &= qdc
\end{aligned}
$$

together with

$$a^N = d^N = \mathbf{1}, \qquad c^N = b^N = 0$$

These latter generate an ideal I. The element $\mathcal{D} = ad - qbc = da - q^{-1}bc$ is central and represents the q-determinant, and if we set it equal to 1, we get the quotiented $\mathrm{Fun}(SL_q(2))/I \equiv \mathcal{F}$.

Furthermore, the following commutative diagram of algebras and algebra homomorphisms was introduced in ref. 6:

$$
\begin{array}{ccc}
\mathcal{A}(C^2) & \xrightarrow{\rho} & \mathcal{A}(C^2) \otimes \mathcal{A}(SL(2; C)) \\
fr\downarrow & & fr\otimes Fr\downarrow \\
\mathcal{A}(C^2) & \xrightarrow{\rho_q} & \mathcal{A}(C_q^2) \otimes \mathcal{A}(SL_q(2; C)) \\
\pi_M\downarrow & & \pi_M\otimes\pi_F\downarrow \\
M_3(C) & \xrightarrow{\rho_F} & M_3(C) \otimes \mathcal{A}(F)
\end{array}
$$

where $\mathcal{A}(.)$ means the algebra of polynomial functions.

At this level, it is important to point out that in order to be a Hopf algebra, $\mathcal{F}$ must be defined for q an *odd primitive* root of unity.

Using the fact that $a^N = \mathbf{1}$ and that

$$ad = \mathbf{1} + qbc$$

we obtain $d = a^{N-1}(\mathbf{1} + qbc)$, so that d (or a) can be eliminated.

The algebra $\mathcal{F}$ can therefore be linearly generated, as a vector space, by the elements $a^\alpha b^\beta c^\gamma$, where $\alpha, \beta, \gamma = 0, 1, 2, \ldots, N - 1$. We see that $\mathcal{F}$ is an associative algebra, of dimension N^3 [7].

2.3. The Quantum Algebra $\mathcal{H}$ and Its Action on $\mathcal{M}_N$

Using the interchange of multiplication and comultiplication by duality, we define the dual $\mathcal{H}$ of $\mathcal{F}$ as a quantum group of the same dimension as $\mathcal{F}$, generated by $H^\alpha X_+^\beta X_-^\gamma$, where $\alpha, \beta, \gamma = 0, 1, 2, \ldots, N - 1$ and X_+, X_-, H are defined by duality by means of the following pairings between generators:

$$
\begin{array}{llll}
\langle H, a\rangle = q, & \langle H, b\rangle = 0, & \langle H, c\rangle = 0, & \langle H, d\rangle = q^2 \\
\langle H^{-1}, a\rangle = q^2, & \langle H^{-1}, b\rangle = 0, & \langle H^{-1}, c\rangle = 0, & \langle H^{-1}, d\rangle = q \\
\langle X_+, a\rangle = 0, & \langle X_+, b\rangle = 1, & \langle X_+, c\rangle = 0, & \langle X_+, d\rangle = 0 \\
\langle X_-, a\rangle = 0, & \langle X_-, b\rangle = 0, & \langle X_-, c\rangle = 1, & \langle X_-, d\rangle = 0
\end{array}
$$

and the relation

$$
\begin{aligned}
H^N &= \mathbf{1} \\
X_+^N &= X_-^N = 0
\end{aligned}
$$

Furthermore, the duality operation is a delicate issue for infinite-dimensional spaces since the convergence is not well defined for the algebra $\mathcal{H}$ (see ref. 8, where the case $N = 3$ is given explicitly). $\mathcal{H}$ acts on the cyclic quantum plane $\mathcal{M}_N$ since its dual $\mathcal{F}$ coacts on it. There are again two possibilities, left or right, but we shall use the left action, which is generally defined as follows. If we denote the right coaction of $\mathcal{F}$ on $\mathcal{M}_N$ as

$$\delta_R(z) = \sum_i z_i \otimes u_i$$

then

$$
\begin{aligned}
X_L(z) &= (Id \otimes \langle X_L, .\rangle) \circ \delta_R(z) \\
&= (Id \otimes \langle X_L, .\rangle)(\Sigma_i z_i \otimes u_i) \\
&= \Sigma_i \langle X_L, u_i\rangle z_i
\end{aligned}
$$

for $z, z_i \in \mathcal{M}_N$, $X_L \in \mathcal{H}$, $u_i \in \mathcal{F}$.

It follows that the action of $\mathcal{H}$ on $\mathcal{M}_N$ is given by the following table:

Left	H	X_+	X_-
$\mathbf{1}$	$\mathbf{1}$	0	0
x	qx	0	y
y	q^2y	x	0

For an arbitrary element of $\mathcal{M}_N$, one find the following expressions:

$$H^L[x^r\, y^s] = q^{(r-s)}\, x^r\, y^s$$

$$X_+^L[x^r\, y^s] = q^r\left(\frac{1 - q^{-2s}}{1 - q^{-2}}\right) x^{r+1}\, y^{s-1}$$

$$X_-^L[x^r\, y^s] = q^s\left(\frac{1 - q^{-2r}}{1 - q^{-2}}\right) x^{r-1}\, y^{s+1}$$

with $r, s = 0, 1, 2, \ldots, N - 1$.

2.4. Reduction of the Algebra $\mathcal{M}_N$ into Indecomposable Representation of $\mathcal{H}$

The generator H always acts as an automorphism; for this reason, in order to study the invariant subspaces of $\mathcal{M}_N$ under the left action of $\mathcal{H}$, we have only to consider the action of X_+ and X_-. Neglecting numerical factors, the action of X_+ and X_- on a given element of $\mathcal{M}_N$ can be written as follows:

$$x^{r+1}y^{s-1} \rightleftharpoons x^r y^s \rightleftharpoons x^{r-1}y^{s+1}$$

where X_- takes us from the left to the right and X_+ from the right to the left.

We verify that under the left action of $\mathcal{H}$ the algebra of $N \times N$ matrices can be decomposed into a direct sum of N subspaces of dimension N according to

$$N_N = \{x^{N-1}, x^{N-2}y, x^{N-3}y^2, x^{N-4}y^3, \ldots, xy^{N-2}, y^{N-1}\}$$
$$N_{N-1} = \{x^{N-2}, x^{N-3}y, x^{N-4}y^2, \ldots, xy^{N-3}, y^{N-2}, x^{N-1}y^{N-1}\}$$
$$N_{N-2} = \{x^{N-3}, x^{N-4}y, x^{N-5}y^2, \ldots, xy^{N-4}, y^{N-3}, x^{N-1}y^{N-2}, x^{N-2}, y^{N-1}\}$$
$$N_{N-3} = \{x^{N-4}, x^{N-5}y, x^{N-6}y^2, \ldots, xy^{N-5}, y^{N-4}, x^{N-1}y^{N-3}, x^{N-2}y^{N-2}, x^{N-3}y^{N-1}\}$$
$$N_{N-4} = \{x^{N-5}, x^{N-6}y, x^{N-7}y^2, \ldots, xy^{N-6}, y^{N-5}, x^{N-1}y^{N-4}, x^{N-2}y^{N-3}, x^{N-3}y^{N-2}, x^{N-4}y^{N-1}\}$$
$$\cdots$$
$$N_2 = \{x, y, x^{N-1}y^2, x^{N-2}y^3, \ldots, x^3y^{N-2}, x^2y^{N-1}\}$$
$$N_1 = \{\mathbf{1}, x^{N-1}y, x^{N-2}y^2, x^{N-3}y^3, \ldots, x^2y^{N-2}, xy^{N-1}\}$$

such that

$$\mathcal{M}_N = N_N \oplus N_{N-1} \oplus \cdots \oplus N_2 \oplus N_1$$

3. THE Z_N-GRADED DIFFERENTIAL GEOMETRY OF Z_N

3.1. General Case

First, let us recall that it is possible to construct a Z_2-graded noncommutative differential geometry of $\mathcal{M}_N$ based on derivations by introducing a set of 1-forms θ^{kl} defined by the following duality relation [2]:

$$\theta^{kl}(e_{mn}) = \delta^{kl}{}_{mn} = \delta^k{}_m\, \delta^l{}_n$$

Then, using the Z_2-graded differential d (and the wedge product), one easily describe the Z_2-graded noncommutative differential complex $(\Omega_{Der}(\mathcal{M}_N); d)$.

Our main aim in this work is precisely to show that $\mathcal{M}_N$ itself, equipped with some well-defined differential d satisfying $d^N = 0$, can be viewed as a Z_N-graded differential complex of the cyclic group Z_N.

For this purpose, let us define a Z_N-*grading* on $\mathcal{M}_N$ such that

$$|\alpha^{rs}| = \text{grading}\ (\alpha^{rs}) = r + s \quad \text{mod}(N)$$

This means that a Z_N-grading equal to 1 is attributed to the fundamental objects x and y, and then the above decomposition of $\mathcal{M}_N$ is naturally equipped with the following Z_N-grading:

$$\begin{aligned} N_1 &\to 0 \\ N_2 &\to 1 \\ N_3 &\to 2 \\ &\cdots \\ N_{N-2} &\to N-3 \\ N_{N-1} &\to N-2 \\ N_N &\to N-1 \end{aligned}$$

Consider the cyclic group of order N, $Z_N = \{\mathbf{1}, y, y^2, y^3, \ldots, y^{N-1}\}$. Therefore, the algebra $C^\infty(Z_N)$ of complex functions on Z_N can be realized as the algebra $\mathcal{M}_N^{\text{diag}} \subset \mathcal{M}_N$ of diagonal complex $N \times N$ matrices.

Starting from $C^\infty(Z_N) \equiv \Omega^0\ (Z_N) = Z_N$, we can build the space of 1-forms $\Omega^1(Z_N)$ by introducing a differential d_x: $\Omega^0 \to \Omega^1$ associated to x and defined by

$$d_x(y^m) = [x, y^m] = (1 - q^{-m})xy^m$$

This means that the subspace $\Omega^1 = x\Omega^0 = \{x, xy, \ldots, xy^{N-1}\}$ constitutes the space of 1-forms.

This differential can be naturally extended to all other subspaces of $\mathcal{M}_N$ such that

$$d_x: \quad \Omega^k \rightarrow \Omega^{k+1}$$

$$d_x(\alpha^{rs}) = x\alpha^{rs} - q^r\alpha^{rs}x = [x, \alpha^{rs}]_q = (1 - q^{r-s})\alpha^{(r+1),s} \tag{4}$$

where the subspace of k-forms is defined by

$$\Omega^k = x^k\Omega^0 = \{x^k, x^k y, \ldots, x^k y^{N-1}\}$$

for $k = 0, 1, \ldots, N - 1$. It is easy to see that the *degree* of the differential forms is given by

$$\text{degree}(\alpha^{rs}) = r \quad \text{mod}(N)$$

and that the wedge product between two arbitrary forms is nothing else than the usual matrix multiplication.

Then the Z_N-graded differential complex $(\Omega(Z_N), d)$ with $d^N = 0$ is completely built with

$$\Omega(Z_N) = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \cdots \oplus \Omega^{N-2} \oplus \Omega^{N-1} \approx M_N(C)$$

Moreover, one can easily verify that the differential d satisfies a q-deformed Leibniz rule:

$$d_x(\alpha^{rs}\alpha^{mn}) = (d_x(\alpha^{rs}))\alpha^{mn} + q^r\alpha^{rs}(d_x(\alpha^{mn}))$$

and that effectively one has $d^N = 0$:

$$\begin{aligned} d_x^N(\omega_p) &= [x, [x, [x, \ldots, [x, \omega_p]_q]_q \cdots]_q \quad (N \text{ times}) \\ &= [x, [x, [x, (x\omega_p - q^p\omega_p x)]_q \cdots]_q \\ &= q^p(1 + q + q^2 + \cdots + q^{N-1})(\cdots) + (x^N\omega_p - \omega_p x^N) \\ &= 0 \end{aligned}$$

3.2. $N = 3$

Let us now consider the case of $Z_3 = \{\mathbf{1}, y, y^2\}$, with

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = y^3, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \qquad y^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & q \end{pmatrix}$$

with $q = e^{2i\pi/3}$.

The algebra $C^\infty(Z_3)$ of complex functions on Z_3 is then identified with the subalgebra $\mathcal{M}_3^{\text{diag}} \subset \mathcal{M}_3$ of diagonal complex 3×3 matrices, where $\mathcal{M}_3$ is generated by;

$$\{\mathbf{1}, x, y, xy, x^2, y^2, x^2y, xy^2, x^2y^2\}$$

with

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

If we attribute a Z_3-grading 1 to x and y, then we have

$$\begin{aligned} \{\mathbf{1}, xy^2, x^2y\} &\to 0 \\ \{x, y, x^2y^2\} &\to 1 \\ \{x^2, y^2, xy\} &\to 2 \end{aligned}$$

From the subspace $\Omega^0 = Z_3$ of 0-forms, we build the two other subspaces of 1- and 2-forms, respectively,

$$\begin{aligned} \Omega^1 &= x\Omega^0 = \{x, xy, xy^2\} \\ \Omega^2 &= x^2\Omega^0 = \{x^2, x^2y, x^2y^2\} \end{aligned}$$

by using the differential d_x: $\Omega^k \leftarrow \Omega^{k+1}$ defined by (4), i.e.,

$$\begin{aligned} d_x(\mathbf{1}) &= 0 \\ d_x(y) &= (1 - q^2)xy \\ d_x(y^2) &= (1 - q)xy^2 \\ d_x(x) &= (1 - q)x^2 \\ d_x(xy) &= 0 \\ d_x(xy^2) &= (1 - q^2)x^2y^2 \\ d_x(x^2) &= (1 - q^2)\mathbf{1} \\ d_x(x^2y) &= (1 - q)y \\ d_x(x^2y^2) &= 0 \end{aligned} \tag{5}$$

Then, the Z_3-graded differential algebra $\Omega(Z_3)$ is given by

$$\Omega(Z_3) = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \approx M_3(C)$$

with

$$\Omega^k = x^k Z_3, \qquad k = 0, 1, 2$$

Finally, using the relations (5), we can easily verify that for arbitrary $\omega_p \in \Omega^p(Z_N)$ and $\omega_q \in \Omega^q(Z_N)$ we have

$$d_x(\omega_p\omega_q) = (d_x\omega_p)\omega_q + q^p\omega_p(d_x\omega_q)$$

and

$$\begin{aligned} d_x^3(\omega_p) &= [x, [x, [x, \omega]_q]_q]_q \\ &= [x, [x, (x\omega - q^p\omega x)]_q]_q \\ &= \cdots \\ &= q^p(1 + q + q^2)x[x, \omega_p x]_q + (x^3\omega - \omega x^3) \\ &= 0 \end{aligned}$$

4. CONCLUSION

Noncommutative differential geometry has become a very important research topic in mathematical physics. In this context, the role of the C^*-algebra of continuous complex functions on an ordinary manifold is played by an abstract *associative*, not necessarily commutative C^*-algebra as analog of functions on noncommutative manifolds. In order to define gauge theories on these noncommutative spaces, we need to define noncommutative differential calculus on them. In fact, several particle physics models have been constructed on noncommutative spaces, for instance, on product spaces like $C^\infty(M) \otimes M_N(C)$, $M_4 \times Z_N$, etc. [2, 9] where M_4 is Minkowski space.

The matrix algebra $\mathcal{M}_N$ is very often used in various fields of physics, and it was shown that its differential geometry is the simplest example of noncommutative differential geometry. In ref. 3 the Wess–Zumino complex of $\mathcal{M}_N$ was constructed. Following the Dubois-Violette approach [2], we developed the noncommutative universal differential algebra of these matrix algebras and presented its decomposition into irreducible components by determining the eigenvalue equations of the associated Laplace–Beltrami operator, with a special interest in the case of $M_3(C)$ [10].

It would be very interesting to study the Z_N-graded differential geometry of some noncommutative spaces. We plan to treat this subject in a future paper in order to describe gauge theories on such spaces.

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